The Mathematical Model: further details

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Hypergraph

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- can be represented by bipartite graph with 2 kind of nodes

Learning

- Delta learning:

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- Variation+selection: add/delete nodes

$$fitness(E_j) = \frac{\sum ||\mathbf{c}|| - ||f_a(\mathbf{c})||}{|E_j|}$$
$$mutate(E_j) \sim \frac{1}{fitness(E_j)}$$

Topology

- Scale-free: power-law in degree-distribution of nodes and edges

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- Centrality:
  - Eigenvector centrality
  - General centrality $\mathbf{c}(\alpha, \beta)$

Eigenvector centrality

Graph	Hypergraph
$\lambda e_i = \sum_j W_{ij} e_j$	$x_i = c_1 \sum_j W_{ij} y_j$ $y_j = c_2 \sum_i W_{ij} x_i$
$\lambda \mathbf{e} = W\mathbf{e}$	$WW^T\mathbf{x} = \lambda\mathbf{x}$ $W^T W\mathbf{y} = \lambda\mathbf{y}$ with $\lambda = c_1 c_2$

General centrality

Graph	Hypergraph
$c_i(\alpha, \beta) = \alpha \underbrace{\sum_j W_{ij}}_{\text{degree}} + \beta \underbrace{\sum_j c_j W_{ij}}_{\text{eig. centr.}}$	$x_i = \alpha_1 \sum_j W_{ij} + \beta_1 \sum_j W_{ij} y_j$ $y_j = \alpha_2 \sum_i W_{ij} + \beta_2 \sum_i W_{ij} x_i$
$\mathbf{c}(\alpha, \beta) = \alpha(I - \beta W)^{-1} W \mathbf{1}$	$\mathbf{x} = \alpha_1 W \mathbf{1} + \beta_1 W \mathbf{y}$ $\mathbf{y} = \alpha_2 W^T \mathbf{1} + \beta_2 W^T \mathbf{x}$

Communication in graph

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  = |communications of individual|:
  $W\mathbf{1}$ = paths of length 1;
  $\beta W^2 \mathbf{1}$= communications of length 2; ...

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- our model: $\beta =$ chance challenge chosen

Communication in hypergraph

$$\mathbf{x} = \alpha_1 \sum_{k=0}^{+\infty} (\beta_1 \beta_2 W W^T)^k W \mathbf{1} + \alpha_2 \beta_1 \sum_{k=0}^{+\infty} (\beta_1 \beta_2)^k (W W^T)^{k+1} \mathbf{1}$$

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- $\alpha_1 = 0; \alpha_2 = 1; \beta_1 = 1 \Rightarrow \beta_2 =$ chance challenge selected;

$$\Rightarrow \mathbf{x} = \sum_{k=0}^{+\infty} (\beta_2)^k (W W^T)^{k+1} \mathbf{1}$$

Communication in hypergraph

- General, $\alpha_1 = \alpha_2 = 1$: $\beta_1 =$ chance edge select challenge; $\beta_2 =$chance node select challenge

$$\mathbf{x} = \underbrace{\sum_{k=0}^{+\infty} (\beta_1 \beta_2 W W^T)^k W \mathbf{1}}_{\text{communications to edges}} + \underbrace{\beta_1 \sum_{k=0}^{+\infty} (\beta_1 \beta_2)^k (W W^T)^{k+1} \mathbf{1}}_{\text{communications to nodes}}$$

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$$\begin{array}{ccc}
k = 0 \text{ in 1st} & W\mathbf{1} & \text{comm. to neighbour edge} \\
k = 0 \text{ in 2nd} & \beta_1 W W^T \mathbf{1} & \text{comm. to neighbour nodes} \\
k = 1 \text{ in 1st} & \beta_1 \beta_2 W W^T W \mathbf{1} & \text{comm. to edges at distance 2} \\
\vdots & \vdots & \vdots
\end{array}$$

Improvements

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- learning: add nodes depending on how much they add: $\beta_2 WW^T W =$ chance communication from node to edge in 1 step

Thanks!

Questions?